A Semi-Definite Programming Solution of the Least Order Dynamic Output Feedback Synthesis Problem

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Abstract

It is shown that the least order dynamic output feedback which stabilizes a given linear time invariant plant can be found via a semi-definite program.

Keywords: Least Order Dynamic Output Feedback; Static Output Feedback; Semi-Definite Programming

1 Introduction

Consider the linear time invariant (LTI) plant Σ,

$$\begin{aligned} \Sigma: \quad \dot{x} &= Ax + Bu, && (1.1) \\ y &= Cx, && (1.2) \end{aligned}$$

where $A \in \mathbf{R}^{n \times n}$, $B \in \mathbf{R}^{n \times m}$, and $C \in \mathbf{R}^{p \times n}$. Let $k \leq n$; represent the class of k-th order stabilizing linear controllers for Σ by Σ_c^k, having the general form,

$$\begin{aligned} \Sigma_c^k: \quad \dot{z} &= A_K z + B_K y, && (1.3) \\ u &= C_K z + D_K y; && (1.4) \end{aligned}$$

$A_K \in \mathbf{R}^{k \times k}$.

Two open problems in control theory are stated as follows:

1. **Static Output Feedback (SOF) Problem**: Find polynomial-time verifiable necessary and sufficient conditions on the triplet (A, B, C) such that Σ_c^0 is nonempty.
2. **Least Order Dynamic Output Feedback (LODOF) Problem**: Find a polynomial time algorithm to determine the least k for which Σ_c^k is nonempty.[1]

Note that the SOF is a special case of the LODOF: if the least k in the LODOF problem turns out to be zero, the corresponding SOF has been solved. In fact, the two problems are equivalent via an embedding procedure. We shall refer to both SOF and LODOF as the OFP (Output Feedback Problem).

The OFP has received considerable attention in systems and control community over the last thirty years [1], [5], [7], [11], [13], [14], [15], [16], [22], [23], [25], [27]; also refer to the surveys [2], [26]. In a recent survey on the state of systems and control, the OFP has been identified as an important open problem in control theory [3]. The purpose of the present paper is to solve the OFP using the machinery of semi-definite programming (SDP).[2]

The notation used is mostly standard. Given two sets S_1 and S_2, $S_1 \backslash S_2$ denotes their difference. The n-dimensional Euclidean space is denoted by $\mathbf{R}^n$. The space of $n \times m$ real matrices is denoted by $\mathbf{R}^{n \times m}$; $\mathcal{S}^n$ is the of space of real $n \times n$ symmetric matrices, and $\mathcal{S}^n_+$ and $\mathcal{S}^n_{++}$ are its positive semi-definite and positive definite subsets. A^T and A^{-1} are the transpose and the inverse of the matrix A, respectively, when the latter exists. $A > B$ and $A \geq B$ designate the positive definiteness and positive semi-definiteness of $A - B$. We shall also use a notion of convergence for matrices. For this purpose we have in mind the metric (dist) on $\mathcal{S}^n$ induced by the Frobenius norm.

Most of the background material and proofs are omitted for brevity.

The rest of the paper is devoted to provide a detailed outline of the proof of the following statement.

Theorem 1.1 *The OFP can be solved as an SDP.*[3]

The idea of the proof is as follows: Starting from the formulation of the OFP in terms of the Lyapunov inequality (leading directly to a Bilinear Matrix Inequality (BMI)), elimination lemma and matrix dilation are

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[1]Moreover, find the corresponding k-th order controller.

[2]It is important to stress that the proposed solution method is based on convex optimization algorithms and is by nature, approximate.

[3]Provided that we are given a threshold for determining when an eigenvalue is declared to be zero.

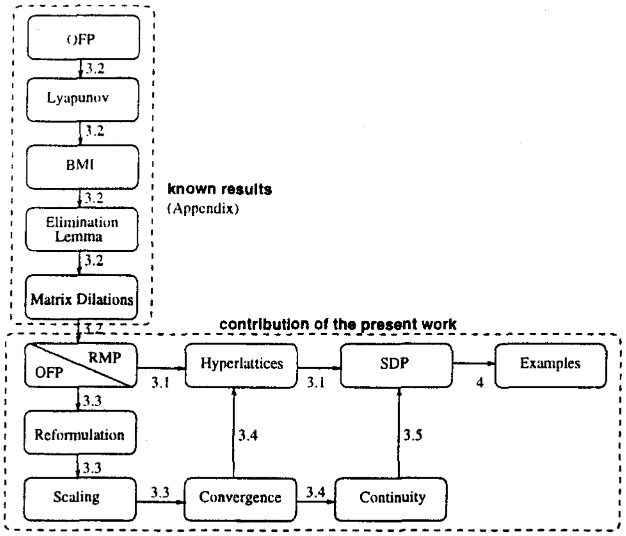

Figure 1: The proof of Theorem 1.1

used to obtain a rank minimization problem (RMP). It is then shown that an RMP whose feasible set enjoys a hyperlattice structure (defined shortly) is solvable as an SDP. The rest of the paper consists of a sequence of propositions and lemmas which show that the feasible set of the RMP which arise from the OFP can be scaled such that it is asymptotically a hyperlattice (for this purpose we introduce ϵ-hyperlattices). Then using a continuity argument, it is argued that given a threshold which is used to declare a number as zero, the OFP can be solved as a SDP. The road map of the proof is shown in Figure 1.

2 Preliminaries

In this section we state few definitions and known facts pertaining the result presented in the paper.

For $A, B \in \mathcal{S}_+^n$, define the matrix intervals

$$\Delta(A,B) := \{X : 0 \leq X \leq A, 0 \leq X \leq B\},$$

and

$$\mathcal{I}(A,B) := \{X : X \leq A, X \leq B\}.$$

Definition 2.1 ([17]) *Let Γ be a nonempty subset of $\mathcal{S}_+^n$. If for all $A, B \in \Gamma$, $\Delta(A,B) \cap \Gamma$ is nonempty, Γ is called a hyperlattice.*

Definition 2.2 ([17]) *Let Γ be a nonempty subset of $\mathcal{S}_+^n$. If there exists a matrix X such that $X \leq Y$ for all $Y \in \Gamma$, then X is the (unique) least element of Γ.*

Proposition 2.1 *Let Γ be a nonempty subset of $\mathcal{S}_+^n$. If Γ admits a least element, that least element has a minimal rank in Γ.*

Proof: Let X be the least element of Γ and assume that there exists $Y \in \Gamma$ such that $\textbf{rank}\ (Y) < \textbf{rank}\ (X)$, i.e., there exists an index j such that $0 = \lambda_j(Y) < \lambda_j(X)$. However, $X \leq Y$ and thus $\lambda_j(X) \leq \lambda_j(Y)$, which is a contradiction. ■

3 Proof of Theorem 1.1

Let us first introduce a convention. Assume that we are given a threshold λ_T, such that if a number $\lambda < \lambda_T$, its value is declared to be zero. Given a parameterized family of $n \times n$ symmetric positive semi-definite matrices $\{A_k\}_{k \geq 1}$, we write

$$\textbf{rank}\ A_k \to m \quad \text{as} \quad k \to \infty$$

whenever $n - m$ eigenvalues of A_k eventually fall below the threshold λ_T, i.e., there exists k_0 such that for all $k \geq k_0$, $\textbf{rank}\ A_k$ is declared to be m.

3.1 RMP with a hyperlattice feasible set

In subsequent sections we assume that the feasible sets of the SDPs or the RMPs are nonempty; note that the feasibility of an SDP, or an RMP whose feasible set is defined by a set of LMIs, can be checked via the interior point methods.

Lemma 3.1 *Let $\Gamma \subseteq \mathcal{S}_+^n$ be nonempty and compact. If Γ is a hyperlattice then,*

$$X^* := \arg\min_{X \in \Gamma} \textbf{Trace}\ \ X,$$

is of minimal rank in Γ.

Definition 3.1 *Let Γ be a nonempty subset of $\mathcal{S}_+^n$. If for a given $\epsilon > 0$, and all $A, B \in \Gamma$, there exists $Z \in \mathcal{I}(A,B)$ such that $dist(Z, \Gamma) \leq \epsilon$, Γ is called an ϵ-hyperlattice.*

Proposition 3.2 *Let Γ_ϵ be a family of nonempty compact ϵ-hyperlattices (parameterized by ϵ). For all $\epsilon > 0$, let*

$$r^* := \min_{X \in \Gamma_\epsilon} \textbf{rank}\ X,$$

and

$$X(\epsilon) := \arg\min_{X \in \Gamma_\epsilon} \textbf{Trace}\ \ X.$$

Then,

$$\textbf{rank}\ X(\epsilon) \to r^* \quad \text{as} \quad \epsilon \to 0.$$

3.2 The OFP to RMP reduction

Lemma 3.3 ([6], [12]) *There exists $\bar{\gamma} > 0$ such that for all $\gamma \geq \bar{\gamma}$, the OFP can be written as the following optimization problem,*

$$\min_{P,Q} \textbf{rank} \begin{bmatrix} P & I \\ I & Q \end{bmatrix} \tag{3.5}$$

$$\begin{bmatrix} P & I \\ I & Q \end{bmatrix} \geq 0, \tag{3.6}$$

$$AP + PA^T < \gamma BB^T, \tag{3.7}$$

$$A^TQ + QA < \gamma C^TC, \tag{3.8}$$

Corollary 3.4 *For all $\mu > 0$, there exists $\bar{\gamma} > 0$ such that for all $\gamma \geq \bar{\gamma}$, the OFP can be written as the following optimization problem,*

$$\min_{P,Q} \textbf{rank} \begin{bmatrix} P & I \\ I & Q \end{bmatrix} \tag{3.9}$$

$$AP + PA^T < \gamma BB^T, \tag{3.10}$$

$$A^TQ + QA < \gamma C^TC, \tag{3.11}$$

$$\begin{bmatrix} P & I \\ I & Q \end{bmatrix} \geq 0, \tag{3.12}$$

$$P \geq \mu I. \tag{3.13}$$

Corollary 3.5 *For all $\mu > 0$, there exists $\gamma > 0$ such that for all $\gamma_1 \geq \mu\gamma$, $\gamma_2 \geq \gamma/\mu$, the OFP can be written as the following optimization problem,*

$$\min_{P,Q} \textbf{rank} \begin{bmatrix} P & I \\ I & Q \end{bmatrix} \tag{3.14}$$

$$AP + PA^T < \gamma_1 BB^T, \tag{3.15}$$

$$A^TQ + QA < \gamma_2 C^TC, \tag{3.16}$$

$$\begin{bmatrix} P & I \\ I & Q \end{bmatrix} \geq 0, \tag{3.17}$$

$$P \geq \mu I. \tag{3.18}$$

Proposition 3.6 *Given the matrices P and Q as solutions to the optimization problem (3.14)-(3.18), the corresponding stabilizing static or least order dynamic output feedback controllers can be found using an LMI.*

3.3 Representation, scaling, and approximation

In this and the next few subsections, we make few observations which are related to the final proof of Theorem 1.1. These results are concerned about establishing that the feasible set of the RMP which corresponds to the OFP approaches a hyperlattice as $\mu \to \infty$ in (3.14)-(3.18) (as long as it remains feasible).

Define the following sets:

$$\mathcal{H}_0 \;:=\; \{X \in \mathcal{S}^{2n} : X = \begin{bmatrix} P & I \\ I & Q \end{bmatrix}\}$$

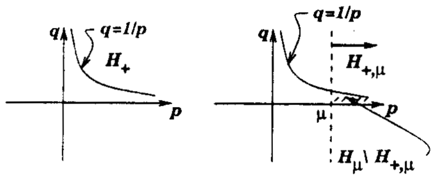

Figure 2: $\mathcal{H}_+, \mathcal{H}_{\mu,+}$

$$\begin{aligned}
\mathcal{H}_+ &:= \{X \in \mathcal{S}^{2n} : X = \begin{bmatrix} P & I \\ I & Q \end{bmatrix} \geq 0\} \\
\mathcal{H}_\mu &:= \{X \in \mathcal{S}^{2n} : X = \begin{bmatrix} P & I \\ I & Q \end{bmatrix}, P \geq \mu I, Q \geq 0\} \\
\mathcal{H}_{+,\mu} &:= \{X \in \mathcal{S}^{2n} : X = \begin{bmatrix} P & I \\ I & Q \end{bmatrix} \geq 0, P \geq \mu I\}
\end{aligned}$$

For $n = 1$, $\mathcal{H}_\mu$ and $\mathcal{H}_{+,\mu}$ are depicted in Figure 2.

Proposition 3.7

$$\mathcal{H}_\mu \to \mathcal{H}_{+,\mu}, \quad as \quad \mu \to \infty. \tag{3.19}$$

Proposition 3.8 *For all $\mu > 0$, there exists $\gamma > 0$ such that for all $\gamma_1 \geq \mu\gamma$, $\gamma_2 \geq \gamma/\mu$, the OFP is equivalent to,*

$$\min_{X} \textbf{rank}\, X \tag{3.20}$$

$$X - L_{\alpha\beta} X L_{\alpha\beta}^T + R_\omega > 0, \tag{3.21}$$

$$X \in \mathcal{H}_{+,\mu} \tag{3.22}$$

where,

$$\begin{aligned}
\omega &= \{\gamma_1, \gamma_2, \alpha, \beta\} \\
J &= \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \\
L_{\alpha\beta} &= \begin{bmatrix} L_1 & L_2 \\ L_3 & L_4 \end{bmatrix} \\
L_1 &= (\alpha I - A)^{-1}(\alpha I + A) \\
L_2 &= L_3 = 0 \\
L_4 &= (\beta I - A^T)^{-1}(\beta I + A^T) \\
R_o &= \begin{bmatrix} R_1 & R_2 \\ R_3 & R_4 \end{bmatrix} \\
R_1 &= \alpha\gamma_1(\alpha I - A)^{-1} BB^T (\alpha I - A)^T \\
R_2 &= R_3 = 0 \\
R_4 &= \beta\gamma_2(\beta I - A)^{-1} C^TC(\beta I - A^T)^{-1} \\
R_\omega &= 2R_0 - J + L_{\alpha\beta} J L_{\alpha\beta}^T.
\end{aligned}$$

Let the linear map $F_\omega : \mathcal{S}^{2n} \to \mathcal{S}^{2n}$ represent the assignment,

$$F_\omega(X) := L_{\alpha\beta} X L_{\alpha\beta}^T - R_\omega \tag{3.23}$$

where $\gamma_1 = \mu\gamma$ and $\gamma_2 = \gamma/\mu$.

Proposition 3.9 *For all $\mu > 0$, $X \in \mathcal{H}_{+,\mu}$, and $\gamma > 0$, $F_\omega(X) \to \mathcal{S}_+^{2n}$, as $\alpha, \beta \to \infty$.*

Proposition 3.10 *For all $X \in \mathcal{H}_0$ and ω,*

$$F_\omega(X) \in \mathcal{H}_0.$$

Let $E : \Re \to \mathcal{S}_{++}^{2n}$ represent a strict monotonically increasing (SMI) matrix function, i.e.,

$$\epsilon_1 < \epsilon_2 \quad \Rightarrow \quad E(\epsilon_1) < E(\epsilon_2).$$

Corollary 3.11 *For all $\mu > 0$ and SMI matrix function E, there exist $\gamma > 0$ and $\epsilon_0 > 0$, such that for all $0 < \epsilon \leq \epsilon_0$, $\gamma_1 \geq \mu\gamma$, and $\gamma_2 \geq \gamma/\mu$, the OFP can be written as the following optimization problem,*

$$\begin{aligned} \min_X \textbf{rank } X \\ X - L_{\alpha\beta} X L_{\alpha\beta}^T + R_{\omega,\epsilon} \geq 0, \\ X \in \mathcal{H}_{+,\mu}, \end{aligned}$$

where

$$R_{\omega,\epsilon} = R_\omega - E(\epsilon).$$

Given E, ω, and ϵ, analogous to (3.23) let us define $F_{\omega,\epsilon} : \mathcal{S}^{2n} \to \mathcal{S}^{2n}$ to represent the assignment,

$$F_{\omega,\epsilon}(X) := L_{\alpha\beta} X L_{\alpha\beta}^T - R_{\omega,\epsilon}, \tag{3.24}$$

where $\gamma_1 = \mu\gamma$ and $\gamma_2 = \gamma/\mu$.

Proposition 3.12 *For all $X \in \mathcal{H}_0$, $F_{\omega,\epsilon} \in \mathcal{H}_0$. Moreover, for all $\mu > 0$, $X \in \mathcal{H}_{+,\mu}$, $\gamma > 0$, and $\epsilon > 0$,*

$$F_{\omega,\epsilon}(X) \to \mathcal{S}_{++}^{2n} \quad as \quad \alpha, \beta \to \infty$$

Proof: Note that $F_{\omega,\epsilon}(X) = F_\omega(X) + E(\epsilon)$ and $F_\omega(X) \to \mathcal{S}_+^{2n}$ as $\alpha, \beta \to \infty$. ■

We now enforce, without loss of generality, the following relationships among the parameters μ, α, and β, and adopt a particular form for the matrix function E,

$$\alpha = \sqrt{\mu}, \quad \beta = \mu^2, \quad \text{and} \quad E(\epsilon) = \begin{bmatrix} \epsilon_1 I & 0 \\ 0 & \epsilon_2 I \end{bmatrix};$$

moreover, since in principle, we would like $\epsilon_1, \epsilon_2 \to 0$, we let

$$\epsilon_1 = \frac{1}{\mu}, \quad \text{and} \quad \epsilon_2 = \frac{1}{\mu^4}.$$

With these relationships enforced we defined L_μ, R_μ and F_μ to stand for $L_{\alpha\beta}$, $R_{\omega,\epsilon}$ and $F_{\omega,\epsilon}$, respectively.[4]

[4] The reader might wonder why we chose these particular functional dependencies among the parameters. The final form of the semi-definite program which we end up solving will shed some light on the rationale for these selections.

Proposition 3.13 *For all $\mu > 0$ and $X \in \mathcal{H}_0$, $F_\mu \in \mathcal{H}_0$. Moreover, as $\mu \to \infty$, for all $X \in \mathcal{H}_{+,\mu}$, $F_\mu(X) \to \mathcal{S}_+^{2n}$.*

Corollary 3.14 *For all $X \in \mathcal{H}_\mu$, $F_\mu(X) \to \mathcal{S}_+^{2n}$.*

Proof: Recall that $\mathcal{H}_\mu \to \mathcal{H}_{\mu,+}$ as $\mu \to \infty$. ■

3.4 The ϵ-hyperlattice structure

Lemma 3.15 *For all $\mu > 0$, let*

$$\Gamma_\mu := \{X \in \mathcal{H}_{+,\mu} : X - L_\mu X L_\mu^T + R_\mu \geq 0\}, \tag{3.25}$$

$$n + k^* := \min_{X \in \Gamma_\mu} \textbf{rank } X,$$

and

$$X(\mu) := \arg \min_{X \in \Gamma_\mu} \textbf{Trace } X.$$

Then

$$\textbf{rank } X(\mu) \to n + k^* \quad as \quad \mu \to \infty.$$

3.5 Back to the original representation

The procedure for solving the OFP using the SDP approach implicitly described above suggests the following algorithm:

1. Input the triplet $A \in \mathbf{R}^{n \times n}$, $B \in \mathbf{R}^{n \times m}$, and $C \in \mathbf{R}^{p \times n}$.
2. Choose the parameters μ and γ.
3. Set $\alpha = \sqrt{\mu}$, $\beta = \mu^2$, and $\epsilon_1 = \frac{1}{\mu}$ and $\epsilon_2 = \frac{1}{\mu^4}$.
4. Solve the following SDP:

$$\min_{P+Q} \textbf{Trace } P + Q \tag{3.26}$$

$$AP + PA^T \leq \mu\gamma BB^T - \frac{\epsilon_1}{2}(\alpha I - A^T - A + \frac{1}{\alpha} AA^T), \tag{3.27}$$

$$A^T Q + QA \leq \frac{\gamma}{\mu} C^T C - \frac{\epsilon_2}{2}(\beta I - A - A^T + \frac{1}{\beta} A^T A), \tag{3.28}$$

$$\begin{bmatrix} P & I \\ I & Q \end{bmatrix} \geq 0, \tag{3.29}$$

$$P \geq \mu I. \tag{3.30}$$

5. Find the rank of the solution X^*. The order of the least order dynamic output feedback which stabilizes the plant is $(\textbf{rank } X^*) - n$.
6. Extract the matrices P and Q from the solution X^*. Construct the matrix $\tilde{P} = \begin{bmatrix} P & \star \\ \star & \star \end{bmatrix}$ such that $\tilde{P}^{-1} = \begin{bmatrix} Q & \star \\ \star & \star \end{bmatrix}$.

7. Solve an LMI for the controller K.

Proposition 3.16 *The solution of the SDP (3.27)-(3.30) depends continuously on μ.*

Corollary 3.17 *For all $\mu > 0$, let*

$$\Gamma_\mu := \{X \in \mathcal{H}_{+,\mu} : X - L_\mu X L_\mu^T + R_\mu \geq 0\}, \quad (3.31)$$

$$n + k^* := \min_{X \in \Gamma_\mu} \mathbf{rank}\ X,$$

and

$$X(\mu) := \arg \min_{X \in \Gamma_\mu} \mathbf{Trace}\ X.$$

Let $\lambda_T > 0$ be such that whenever $\lambda_i(X(\mu)) \leq \lambda_T$, its value is declared to be zero. Then there exists $\mu_0 > 0$ such that for all $\mu \geq \mu_0$,

$$\mathbf{rank}\ X(\mu) = n + k^*.$$

Proof: $X(\mu)$ depends continuously on μ and so does its set of eigenvalues. Since **rank** $X(\mu) \to n+k^*$, given the threshold λ_T, there exists $\mu_0 > 0$ such that for all $\mu \geq \mu_0$, $n - k^*$ eigenvalues of $X(\mu)$ will be less than λ_T for all $\mu \geq \mu_0$ ∎

Note that in practice λ_T does not need to be chosen apriori. An obvious rule of thumb for its use is to proceed solving the OFP as the SDP (3.27)-(3.30) for some large value of μ and an appropriate value of γ, and then by inspection *choose* λ_T and determine the rank.[5]

4 Examples

The following examples were solved using the LMI-tool; in order to avoid numerical problems, we avoided choosing μ to be too large.

1. The data for our first example is as follows:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix},$$
$$C = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}.$$

It is known that a second order controller is a minimum order dynamic output feedback which stabilizes this plant.

The eigenvalues of the solution using the SDP formulation described above were found to be,

$$\lambda_1 = 1.1327e-13, \lambda_2 = 8.5827e-04, , \lambda_3 = 0.7460,$$
$$\lambda_4 = 1.1806, \lambda_5 = 1.0000e+03, \lambda_6 = 1.0011e+03,$$
$$\lambda_7 = 1.9130e+03, \lambda_8 = 2.7384e+03.$$

By inspection we declare the rank of the matrix X to be 6; thus a second order controller is in fact the least order dynamic output feedback which stabilizes this plant.

2. The data of our second example is as follows

$$A = \begin{bmatrix} 1.0000 & 1.0050 \\ -1.0050 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

The eigenvalues of the solution were found to be,

$$\lambda_1 = 2.8126e-07, \lambda_2 = 2.8102e-04,$$
$$\lambda_3 = 1.0000e+0, \lambda_4 = 5.7876e+04.$$

Again, by inspection, we declare the rank of the matrix X^* to be two; thus a static output feedback can in fact stabilize this plant.

3. The data of our third example is,

$$A = \begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.0100 & 0.0024 & -4.0208 \\ 0.1002 & 0.3681 & -0.7070 & 1.4200 \\ 0 & 0 & 1.0000 & 0 \end{bmatrix},$$
$$B = \begin{bmatrix} 0.4422 & 0.1761 \\ 3.5446 & -7.5922 \\ -5.5200 & 4.4900 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}.$$

The eigenvalues of the solution were found to be,

$$\lambda_1 = 7.3784e-14, \lambda_2 = 3.2675e-09, \lambda_3 = 5.4909e-04,$$
$$\lambda_4 = 3.500e-3, \lambda_5 = 1.0000e+03, \lambda_6 = 1.0000e+03,$$
$$\lambda_7 = 1.0000e+03, \lambda_8 = 1.7677e+03.$$

By inspection, we declare the rank of the solution to be four, and thus conclude that a static output feedback can stabilize this plant.[6]

5 Conclusion

It is shown that the solution of the least order dynamic output feedback synthesis can be obtained via a semi-definite program.

[5] We provide a result pertaining to the apriori selection of λ_T in the future work.

[6] Note that for these examples the corresponding stabilizing controller can be found via an LMI.

References

[1] B. D. O. Anderson and R. W. Scott. Output feedback stabilization- solution by algebraic geometry methods. *Proceeding of IEEE*, 65:849-861, 1977.

[2] D. S. Bernstein. Some open problems in matrix theory arising in linear systems and control. *Linear Algebra and Its Applications*, 162-164:409–432, 1992.

[3] V. Blondel, M. Gevers, and A. Lindquist. Survey on the state of systems and control. *European Journal of Control*, 1:5–23, 1995.

[4] S. P. Boyd, L. EL Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory.* SIAM, Philadelphia, 1994.

[5] C. I. Byrnes and B. D. O. Anderson. Output feedback and generic stabilizability. *SIAM Journal of Control and Optimization*, 11:362–380, 1984.

[6] L. El Ghaoui and P. Gahinet. Rank minimization under LMI constraints: a framework for output 1

feedback problems. In *Proceedings of the European Control Conference*, July 1993.

[7] M. Fu and Z. Q. Luo. Computational complexity of a problem arising in fixed order output feedback design. *Systems and Control Letters*, 30:209–215, 1997.

[8] J. C. Geromel, C. C. de Souza, and R. E. Skelton. Static Output Feedback Controllers: Stability and Convexity. *IEEE Transactions on Automatic Control*, 43:120–125, 1998.

[9] L. El Ghaoui, R. Nikoukhah, and F. Delebecque. LMITOOL: an package for lmi optimization. In *Proceedings of the IEEE Conference on Decision and Control*, 1995.

[10] K. M. Grigoriadis and R. E. Skelton. Fixed-order control design for LMI control problems using alternating projection method. In *Proceedings of the IEEE Conference on Decision and Control*, 1994.

[11] D. C. Hyland and D. S. Berstein. The optimal projection equations for fixed order dynamic compensation. *IEEE Transactions on Automatic Control*, 29:1034–1037, 1984.

[12] T. Iwasaki and R. E. Skelton. A complete solution to the general H_∞ control problem: LMI existence conditions and state space formulas. In *Proceedings of the American Control Conference*, 1993.

[13] H. Kimura. Pole assignment by gain output feedback. *IEEE Transactions on Automatic Control*, 20:509–516, 1975.

[14] V. Kucera and C. de Souza. A necessary and sufficient conditon for output feedback stabilizability. *Automatica*, 31:1357–1359, 1995.

[15] J. Leventides and N. Karcanias. Global asymptotic linearization of the pole placement map: a closed form solution for the constant output feedback problem. *Automatica*, 31:1303–1309, 1995.

[16] Linnemann. A class of single-input single-output systems stabilizable by reduced-order controllers. *Systems and Control Letters*, 11:27–32, 1988.

[17] M. Mesbahi and G. P. Papavassilopoulos. On the rank minimization problem over a positive semidefinite linear matrix inequality. *IEEE Transactions on Automatic Control*, 42:239–243, 1997.

[18] M. Mesbahi, M. G. Safonov, and G. P. Papavassilopoulos. Bilinearity and complementarity in robust control. In *Recent Advances on LMI Approach in Control.* SIAM, Philadelphia, 1999.

[19] Y. Nesterov and A. Nemirovskii. *Interior-Point Polynomial Algorithms in Convex Programming.* SIAM, Philadelphia, 1994.

[20] A. Packard, K. Zhou, P. Pandey, and G. Becker. A collection of robust control problems leading to LMI's. In *Proceedings of the IEEE Conference on Decision and Control*, 1991.

[21] A. Packard, K. Zhou, P. Pandey, and J. Leonhardson. Optimal, constant I/O similarity scaling for full-information and state-feedback control problems. *Systems and Control Letters*, 19:271–280, 1992.

[22] J. Rosenthal. Pole assignment by real output feedback. *SIAM Journal of Control and Optimization*, 30:203–211, 1992.

[23] J. Rosenthal. On dynamic feedback compensation and compactification of systems. *SIAM Journal of Control and Optimization*, 32:279–296, 1994.

[24] M. G. Safonov, K. C. Goh, and J. Ly. Control system synthesis via bilinear matrix inequalities. In *Proceedings of the American Control Conference*, pages 45–49, Baltimore, MD, June 29–July 1, 1994. IEEE Press, New York.

[25] M. C. Smith. On the minimal order stabilization of single loop plants. *Systems and Control Letters*, 7:39–40, 1986.

[26] V. L. Syrmos, C. Abdallah, P. Dorato, and K. Grigoriadis. Static output feedback: A survey. *Automatica*, 33:125–137, 1997.

[27] X. Wang. Pole placement by static output feedback. *Journal of Mathematical Systems, Estimation, and Control*, 2:205–218, 1992.